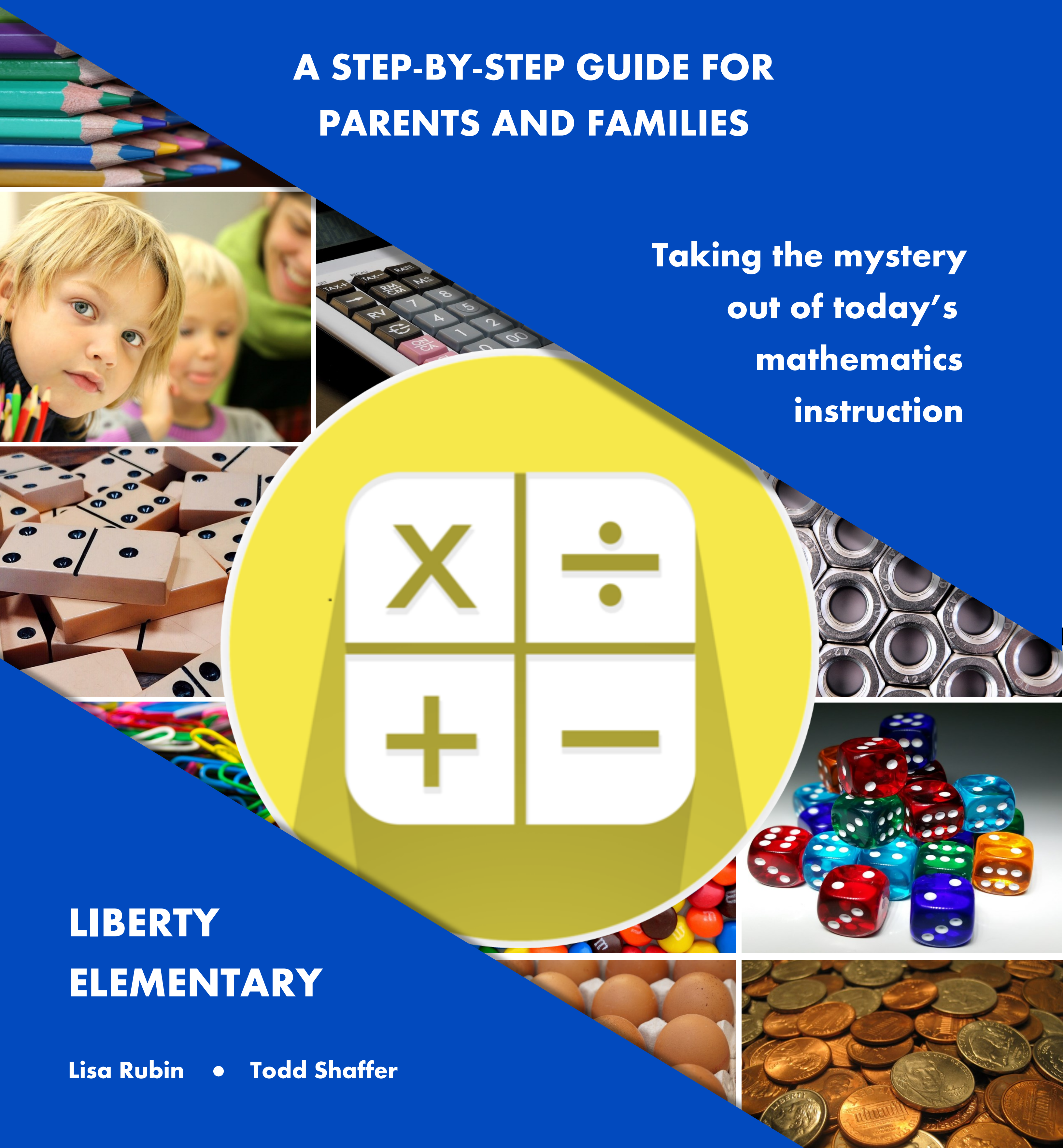
MATHEMATICAL STRATEGIES MADE SIMPLE

A STEP-BY-STEP GUIDE FOR PARENTS AND FAMILIES

Taking the mystery out of today's mathematics instruction



LIBERTY ELEMENTARY

Lisa Rubin • Todd Shaffer

MATHEMATICAL STRATEGIES MADE SIMPLE

A STEP-BY-STEP GUIDE FOR PARENTS AND FAMILIES

This guidebook is dedicated to every adult who has ever said,
"This is not how I learned to do math when I was in school,"
which is pretty much ***all*** of us.

Many of these strategies may be unfamiliar to you.
They were unfamiliar to us when we first saw them as well.

Today's students must be able to do more than just find the answer.
Anyone with a smart phone can find the answer in a few seconds.

Today's students must deeply understand number concepts and number relationships, working flexibly with mathematics to solve complex problems. Having taught children traditional algorithms in the past, and having since taught students these strategies, we believe in these methods as a means of helping all students truly understand the mathematics behind the work they are doing.

Bottom line: We want our students to love math as much as we do.
We want our students to deeply understand math and be able to explain it to others.

We no longer want to teach math how we were taught.
We want to teach it even better.

Thank you for taking time to explore this book. It offers brief, but detailed explanations and illustrations of math strategies for addition, subtraction, multiplication and division. We sincerely hope it will be a valuable resource for you throughout your child's elementary school career.

Lisa Rubin
Todd Shaffer

Liberty Elementary

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ADDITION INTRODUCTION

Learning addition is truly about much more than finding the answer. By using strategies like the ones in this book, your child will learn about concepts such as the relative values of numbers, how to decompose numbers into smaller parts, the values of digits within numbers, the structure of the place value system, and why regrouping works. This learning makes students more confident, fluent and flexible in their mathematical reasoning.

Structures of addition problems

Most addition problems simply involve combining two quantities (parts) together to find the **total, or sum**.

Amy read 54 pages yesterday and 39 more today.

How many pages did she read in all?

54 + 39 = **93** pages total

We add:		
	Part	54
	+ Part	+ 39
	= Total	93

In some cases, problems may sound like subtraction, but the **total** is what is missing.

Gerald used 11 of his stamps mailing birthday invitations. He has 14 stamps left. How many stamps did he begin with?

$\boxed{} - 11 = 14$

We know:
The part taken away = 11
The part left = 14
What we don't know:
the total.

→

So, we ***add*** the parts together.		
	Part	11
	+ Part	+ 14
	= Total	25

11 + 14 = 25, so $\boxed{25}$ - 11 = 14. Gerald began with **25** stamps total.

The following pages provide examples and step-by-step instructions for how to perform **5** different addition strategies your child will likely use in elementary school to help build his or her number sense and mathematical flexibility *while* solving addition problems. The traditional standard algorithm is also included, which is taught after students have a strong understanding of addition and the underlying numeracy concepts.

Base Ten Blocks Without Regrouping

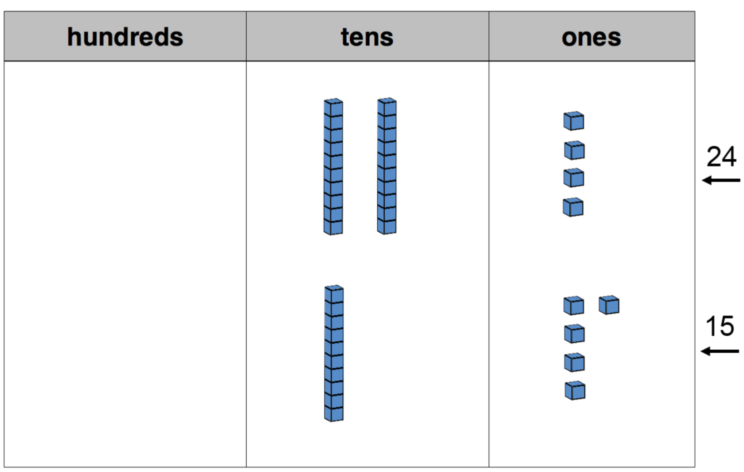
Students build or draw base ten models (blue blocks below) to represent the numbers being added (**addends**), and then combine the quantities to find the total, called the **sum**. Students start with no regrouping, where the total ones will not create a group of 10, or the total tens will not create a group of 100.

Solve: 24 + 15

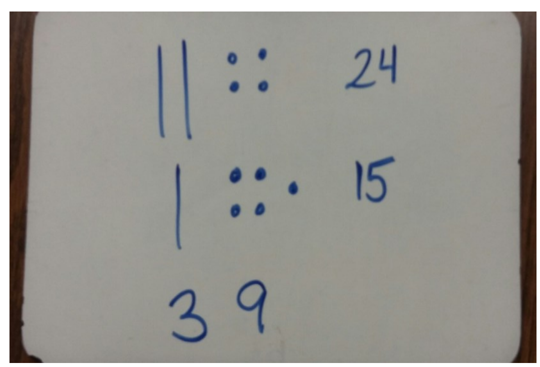
In Context: Billy scored 24 points on Friday and another 15 points on Saturday in his basketball tournament. How many points did he score in both games combined?

Steps:

1. Show 24 by building or drawing 2 tens (20) in the tens column and 4 ones (4) in the ones column.
2. Show 15 by building or drawing 1 ten (10) in the tens column and 5 ones (5) in the ones column.
3. Add the ones together: 4 + 5 = 9. Add the tens together: 20 + 10 = 30.
4. Add the tens and ones together: 30 + 9 = **39**.   **Answer: 39**



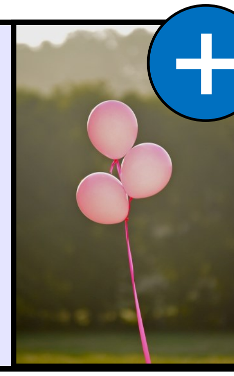
Here is how a student might represent it:



Tip:

Students may be observed adding the tens first, before adding the ones. This can yield the correct answer, but remind students to watch for possible regrouping, where 10 ones will create a ten, or where 10 tens will create a group of one hundred. As students move closer to the traditional addition algorithm, they will want to get in a habit of working right to left, adding the ones first, then the tens, and so on.

Base Ten Block Addition With Regrouping



Students build or draw base ten models (blue and gold blocks below) to represent the numbers to be added, then combine the quantities find the sum. Once students are proficient at solving problems without regrouping, they solve problems where the total ones are enough to create a group of ten, and/or the total tens are enough to create a group of one hundred. Students show these exchanges by grouping 10 ones to make a ten rod, or 10 ten rods to make a one hundred block (called a "flat").

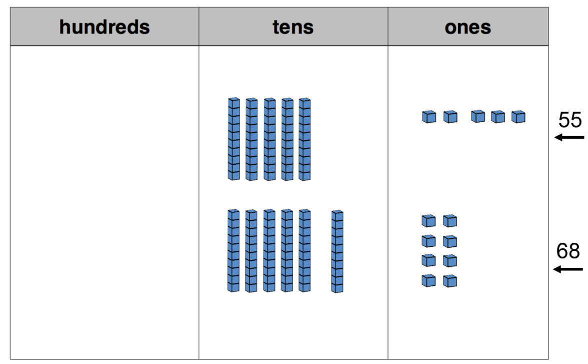
Solve: 55 + 68

In Context: Jan walked 55 laps at Relay for Life. Sara walked 68 laps. How many total laps did the girls walk for their team?

Steps:

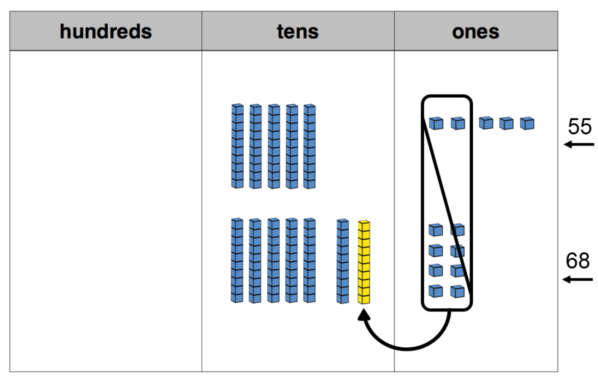
1. Show 55 by building or drawing 5 tens (50) in the tens column and 5 ones (5) in the ones column.

2. Show 68 by building or drawing 6 tens (60) in the tens column and 8 ones (8) in the ones column.



3. Add the ones together.
   5 + 8 = 13

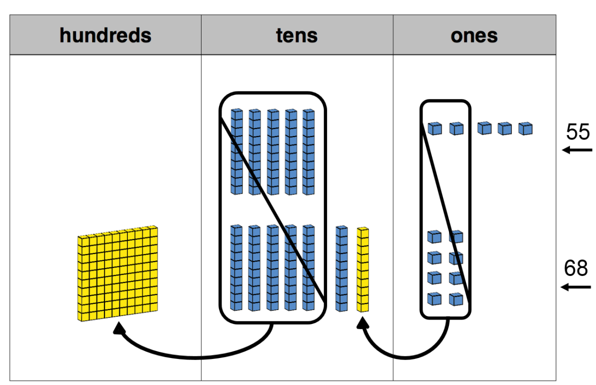
4. Circle 10 of the ones blocks to form an additional tens rod in the tens column.
   After creating a new ten, cross out the 10 ones that were circled.



5. Add the tens together.
   50 + 60 + 10 = 120

6. Circle 10 of the tens rods to form an additional hundreds flat in the hundreds column.
   After creating a new hundred, cross out the 10 tens that were circled.

7. Add 100 + 20 + 3 = **123.** **55 + 68 = 123.**



Adding Using Number Lines

Students draw number lines. The numbers to be added are **decomposed** (broken apart) and shown as jumps along the line. Students can start out at one of the two numbers to be added, start out at a part of one of the numbers, or start out at 0. Once all parts of the numbers have been added, the stopping point is the total.

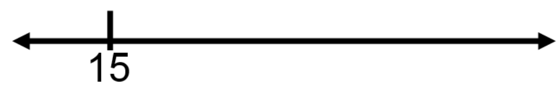
Solve: 15 + 14

In Context: Brittany earned 15 points on a video game on Monday. On Tuesday, she earned 14 points. How many points did she earn in two days?

This is one way to solve this problem with a number line:

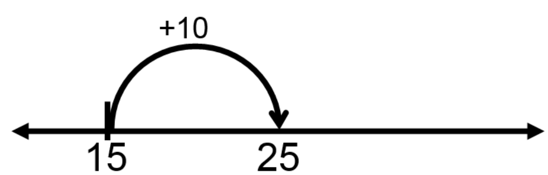
Steps:

1. Draw a blank number line and mark a starting number. Let's start out at 15.



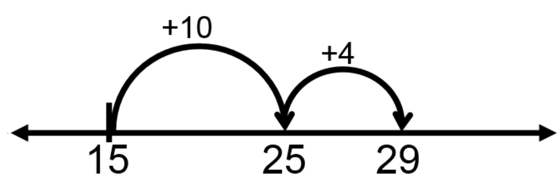
2. From 15, jump ahead 14 more. To make this easier, decompose (split up) the 14 into a jump of 10 and a jump of 4.

3. Draw and label a jump forward of 10 on the number line. Land on 25.



4. Draw and label a jump forward of 4 on the number line. Land on **29.**

Answer: 15 + 14 = 29



In the above problem, Add 15 +14 by starting at 15 and then adding a 10 and a 4. This is just one path for this problem. There are many other correct ways to solve this problem on a number line. **In fact, there is typically *always* more than one path to the solution, and any efficient and accurate path is acceptable.**

Turn to the following page to see the above problem solved several more ways using number lines.

Number Lines (continued)

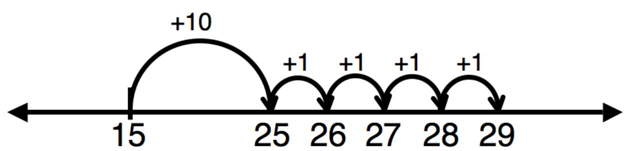
Solve: 15 + 14

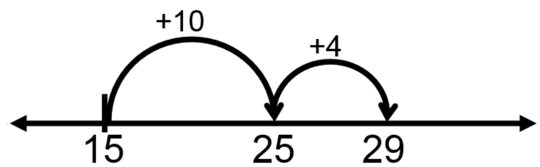
In Context: Brittany earned 15 points on a video game on Monday. On Tuesday, she earned 14 points. How many points did she earn?

Here is the same problem, 15 + 14, solved <u>seven</u> correct ways:

Starting at 15, then adding 14 broken down two different ways:

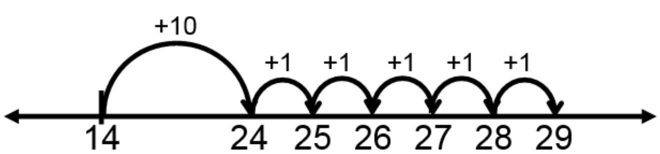
15 + 10 + 1 + 1 + 1 + 1 or 15 + 10 + 4

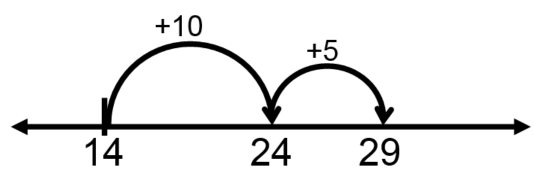




Starting at 14, then adding 15 broken down two different ways:

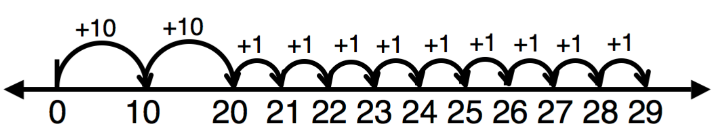
14 + 10 + 1 + 1 + 1 + 1 or 14 + 10 + 5

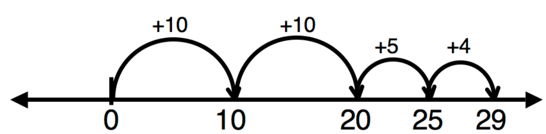




Decomposing (breaking apart) 15 <u>and</u> 14, and adding the parts starting at <u>zero</u>:

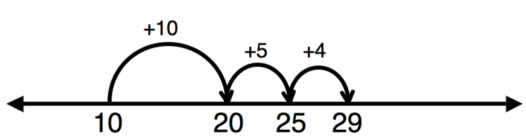
10 + 10 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 or 10 + 10 + 5 + 4





Decomposing (breaking apart) 15 <u>and</u> 14, and adding them starting at <u>one of the tens</u>:

10 + 10 + 5 + 4



Tips:

- Remember that this approach is not about solving efficiently at first. It is about deepening students' understanding of how to decompose numbers and add fluently and flexibly.
- Starting at the larger of the two addends (numbers to be added) is often easier.
- Encourage students who may be adding with only jumps of one to begin taking larger jumps as they show readiness. With experience, students can combine numbers with more efficient and clever combinations.

Number Lines With Friendly Numbers

Students draw a number line. The number to be added is broken apart and represented as jumps along the line. Students can make steps easier by adding friendly numbers (using jumps designed by the student to land on round numbers, such as 47 + 3 = 50), or by chunking numbers into any smaller parts that are easier to add quickly and efficiently using mental math.

Example #1: **Solve:** 35 + 8

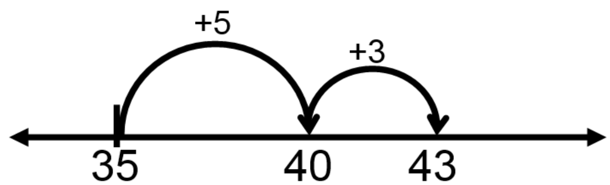
In Context: Juan and Danny have a dog walking business. Juan earned $35. Danny worked less, so he earned just $8. How much money did they earn together?

To add 35 + 8, we can split the 8 into 5 and 3 to make it easier to cross over 40:

35 + 8 is the same as 35 + 5 + 3

Now we can add 35 + 5 in order to land neatly on 40, and then add the remaining 3.

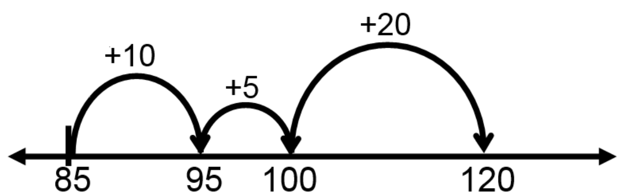
35 + 8 = 35 + 5 + 3 = **43** **Answer: 43**



Example #2: 85 + 35

Start at the larger number, 85. A great approach here is to split the 35 into jumps of 10, 5 and 20, in order to cross more easily over 100 as shown.

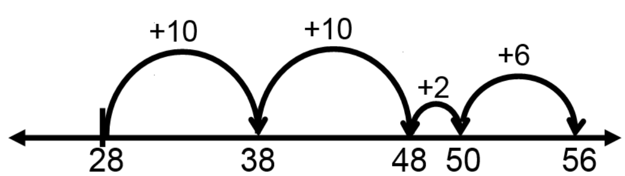
Answer: 120



Example #3: 28 + 28

Start at 28. A good strategy here is to split the other 28 into 10 + 10 + 2 + 6. Then you can jump ahead two friendly jumps of 10, followed by a 2, in order to land on 50. Finally, add the remaining 6.

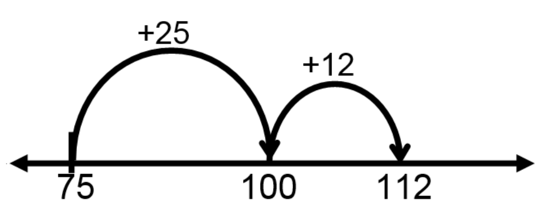
Answer: 56



Example #4: 75 + 37

Students may recognize that 75 + 25 = 100, so it helps to split the 37 into 25 and 12 to create two easy jumps.

Answer: 112



Tips:

- Students are encouraged to draw larger "jumps" to show larger amounts such as +10, and draw smaller "jumps" for smaller amounts such as +2, but the sizes of jumps do not need to be perfectly proportional.
- Students may use friendlier numbers and/or larger jumps, based on their own readiness.

Using Expanded Form to Add

Students can use **expanded form** to break larger numbers into smaller round numbers that are easier to mentally add or subtract. This also helps to develop mental math skills.

The **expanded form** of a number is the value of each digit written out separately in an addition expression. Example: The expanded form of 347 is ***300 + 40 + 7***.

Solve: 65 + 27

In Context: At 9:00 AM, the temperature was 65 degrees Fahrenheit. By 2:00 PM, the temperature had increased by 27 degrees. What was the temperature at 2:00 PM?

Steps:

1. Find the expanded forms of 65 and 27.
   65 can be broken into 60 + 5
   27 can be broken into 20 + 7

65 + 27

65 = 60 + 5   27 = 20 + 7

2. Add the tens from both numbers.
   60 + 20 = 80
3. Add the ones from both numbers.
   5 + 7 = 12

65 = **60** + **5**   27 = **20** + **7**

Tens → **60** + **20** = **80**

Ones → **5** + **7** = **12**

4. Add the total tens and ones together to find the sum.
   Answer: 80 + 12 = **92**.

80 + **12** = **92**

There is not one "right way" to write down the steps taken in this method. Students can organize the steps however they like.

Here, the expanded forms of the numbers are stacked in columns for easy addition.

56 + 24

56 = 50 + 6
24 = 20 + 4
70 + 10 = **80**

In this example, the expanded forms are organized with the hundreds first, then the tens, and then the ones in an expression.

264 + 315 =

(hundreds)		(tens)		(ones)	
200 + 300	+	60 + 10	+	4 + 5	=
500	+	70	+	9	=

579

Tip: As with number lines, remember that this approach is not about solving efficiently at first. It is about helping students understand place value and increasing mathematical flexibility.

Standard Algorithm Without Regrouping

The standard algorithm is used when students have a deep understanding of place value. These examples have no **regrouping** (or what many adults called "carrying the one" when we were growing up in school).

Solve: 123 + 245

In Context: A third grade class donated 123 books to a book sale. A fourth grade class donated 245 books to the book sale. How many books were donated to the book sale?

Steps:

1. Stack the two numbers neatly above one another.
   Line up digits in the ones, tens, and hundreds places under the corresponding place values.

$$\begin{array}{r} 123 \\ +245 \\ \hline \end{array}$$

2. Add the digits in the ones place. 3 + 5 = 8
3. Add the digits in the tens place. 2 + 4 = 6 (20 + 40 = 60)
4. Add the digits in the hundreds place. 1 + 2 = 3 (100 + 200 = 300)

$$\begin{array}{r} 123 \\ +245 \\ \hline 368 \end{array}$$

Answer: 368

Tips:

- Remind children of the values of the digits. For instance, after solving the above example,
  Ask: What is the value of the 6 in your answer? (60)
  How did we get 60? (adding 20 + 40)

- Sometimes a student may incorrectly stack the numbers, such as in this example:

$$\begin{array}{l} \ 243 \\ +25 \\ \hline \end{array}$$

This may reveal that a child does not yet have a clear and complete understanding of place values. If this occurs, it may be time to revisit expanded form, base 10 models, or other concrete representations.

Standard Algorithm With Regrouping

The standard algorithm is used when students have a deep understanding of place value. While it does not show the true value of the digits, it can be an efficient strategy for students who have a solid background. Once regrouping is involved, it becomes even more crucial that students know that ten ones make 10, ten tens make 100, etc.

Solve: $769 + $165

In Context: Malcolm saved his money for several months. When he went to an auction, he was able to buy the go-cart he had dreamed of for $769. He still had $165 left. How much money had he saved <u>before</u> going to the auction?

Steps:

1. Stack the two numbers neatly above one another. Take care to line up the digits in the ones, tens, and hundreds places under the corresponding place values.

$$\begin{array}{r} 769 \\ +\,165 \\ \hline \end{array}$$

2. Add the digits in the ones place. 9 + 5 = 14 ones
3. Don't write 14 in the ones place below the line. Regroup 10 ones to make a ten. Write the 1 above the tens column (over the 6 as shown in red). Then write the remaining 4 ones below the line in the ones place.

$$\begin{array}{r} {\scriptstyle 1} \\ 769 \\ +\,165 \\ \hline 4 \end{array}$$

4. Add the digits in the tens place. 6 + 6 + 1 = 13 tens
5. Don't write 13 in the tens place below the line. Regroup 10 tens to make a hundred. Write the 1 above the hundreds column (over the 7 as shown in green). Then write the remaining 3 tens below the line in the tens place.

$$\begin{array}{r} {\scriptstyle 1\ 1} \\ 769 \\ +\,165 \\ \hline 34 \end{array}$$

6. Add the digits in the hundreds place. 1 + 7 + 1 = 9 hundreds.

   Write the 9 hundreds below the line in the hundreds place.

   Answer: 934

$$\begin{array}{r} {\scriptstyle 1\ 1} \\ 769 \\ +\,165 \\ \hline 934 \end{array}$$

Tip:

Remind your child to only regroup when there is at least ten in any given place value. Some problems may involve regrouping only the tens, only the hundreds, or neither.

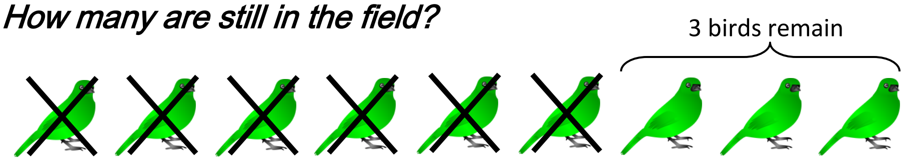
SUBTRACTION INTRODUCTION

When we think of subtraction, we mostly think immediately of "taking from."
Here are three primary ways that subtraction is used to solve problems?

1. Take From

In many subtraction problems, start with a total, then take a part from it, leaving behind the other part.

There were 9 birds in the field. 6 flew away.
How many are still in the field?

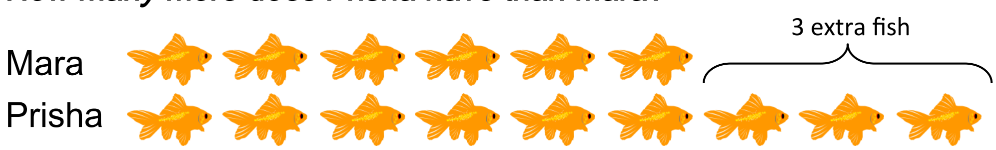


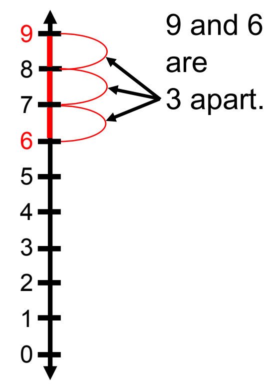
9	Total
− 6	- Part taken
3	= Part left

2. Compare

Subtraction is used to find the difference between two numbers,
or in other words, how much more (or less) one number is than another.

Mara has 6 goldfish. Prisha has 9.
How many more does Prisha have than Mara?





9 - 6 = 3 because 9 is 3 more than 6. The difference between 9 and 6 is 3.

3. Missing Addend

A third reason why we subtract is to find the missing <u>part</u> (addend) in an addition sentence.

Barry had $6 to begin with. He earned more money, and now he has $9. How much did he earn?

6 + $\boxed{?}$ = 9

We usually subtract: Total − Known Part = Missing Part → We know: The total = $9, The part Barry had = $6 → So, we subtract: Total 9 − Known Part - 6 = Missing Part $\boxed{3}$

Barry earned $3 more. 9 - 6 = 3, so 6 + 3 = 9

Base Ten Blocks Without Renaming

Students build or draw models to represent subtraction using a “take from” approach using blocks or pictures of base ten blocks. Students start with no renaming (“borrowing”). The starting number (**minuend**) has enough ones and tens to give away during the subtraction. Students take away or cross out the amount to be subtracted in order to find the amount left, called the **difference**.

Solve: 68 — 42

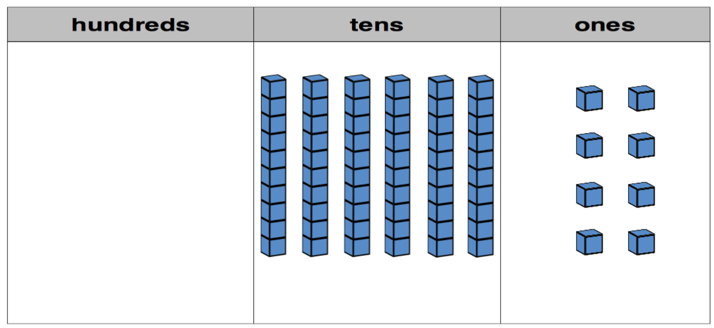
minuend subtrahend

In Context: Marta has gotten an impressive 68 base hits this season. She had 42 hits for her team last year. How many more hits has she gotten this year compared to last year?

Steps:

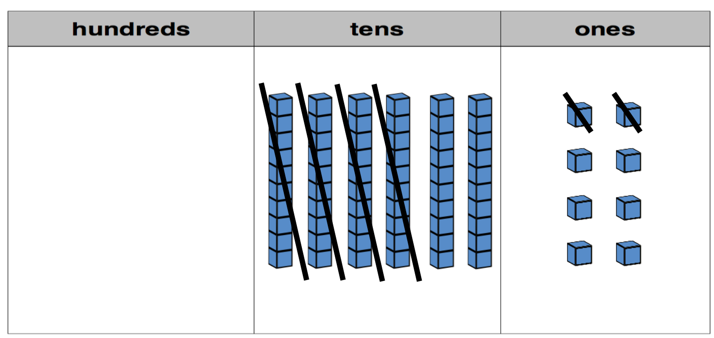
1. Build or draw the starting number (minuend). Show 68 by building or drawing 6 tens (60) in the tens column and building or drawing 8 ones (8) in the ones column.

Students do not need to build or draw the 42.



2. Take away the 42 (subtrahend).
   Remove/cross out 2 ones (2).
   Remove/cross out 4 tens (40).
3. Count the number of tens that are left.
   (2 tens = 20)
4. Count the number of ones that are left.
   (6 ones = 6)

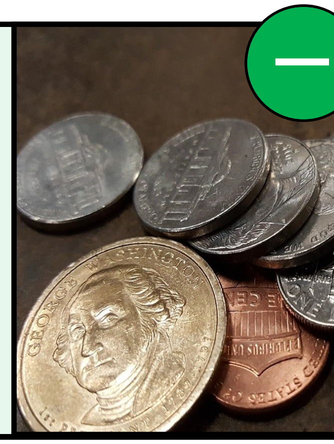
Answer = 68 - 42 = 26



Tip:

- Students may be observed subtracting the tens first, before subtracting the ones. This can yield the correct answer when no renaming is needed, but not when there are not enough of a given place to take away. Students will want to quickly form a habit of working right to left, subtracting the ones first, then the tens, etc.

Base Ten Blocks With Renaming



Students build or draw models to represent subtraction using a "take from" approach using blocks or pictures of base ten blocks. Here, students solve problems where the number of ones or tens in the starting number (minuend) is less than the ones or tens in the number to be taken away (subtrahend). Students must "borrow" by replacing a ten rod to make 10 ones, or a 100 block to make 10 tens, and so on. Students then take away or cross out the amount to be taken away (subtrahend) in order to find the amount left (difference).

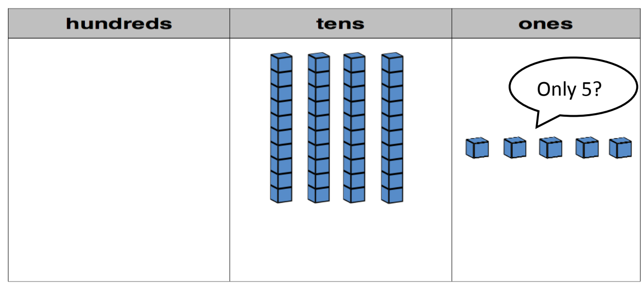
Solve: 45 — 27

minuend subtrahend

In Context: Mary flipped a coin 45 times. The coin came up heads 27 times. How many times did her coin show tails?

Steps:

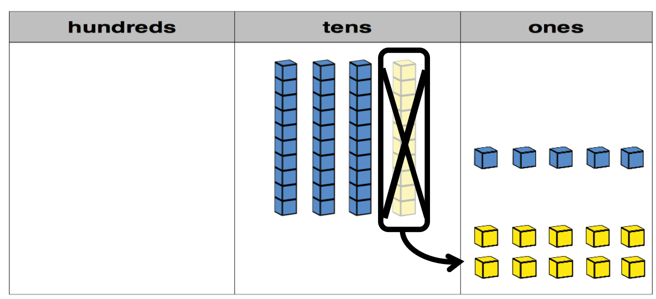
1. Build or draw the starting number (minuend). Show 45 by building or drawing 4 tens (40) in the tens column and building or drawing 5 ones (5) in the ones column.
   Students do not need to build the 27.



2. There are not enough ones in 45 to remove or cross out 7 ones.

3. Rename a ten. Move 1 ten (10) from the tens column and replace it with 10 ones (10) in the ones column.

   Notice that there is still a value of 45 cubes shown at this point.
   3 tens (30) and 15 ones (15) is still 45.

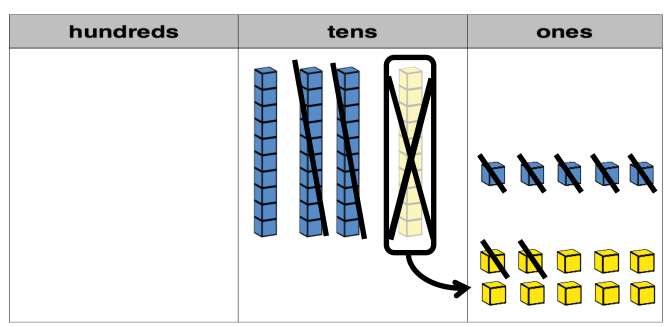


4. Now there are 15 ones, so you can subtract the 7 ones in 27.

   Begin to take away the 27 (subtrahend).

   Remove/cross out 7 ones (7).

   Remove/cross out 2 tens (20).



5. Count the number of tens that are left (1 ten = 10).
6. Count the number of ones that are left (8 ones = 8).

Answer = 45 - 27 = 18

Tip: Students must work right to left, subtracting the ones first, then the tens, etc.

Number Lines to Count Back

Students draw a number line and then represent subtraction of the decomposed parts of the numbers as **backwards** jumps along the line. Once all parts of the numbers have been subtracted, the stopping point is the answer (called the difference).

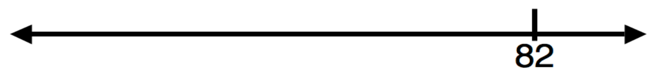
Solve: 82 - 24

In Context: There were 82 students in the fourth grade. 24 of the students did their homework for extra credit. How many students will not receive extra credit?

Here is one way to solve this problem with a number line.

Steps:

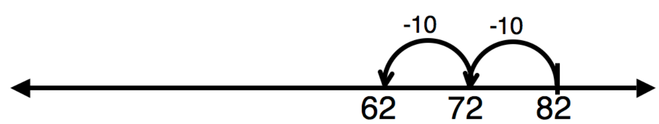
1. Draw a blank number line and place the starting number (82) on the right, leaving space to the left to count backwards.



2. We need to jump backward 24. There are many ways to decompose the 24. One way is to subtract 24 by going backwards 2 jumps of 10 and then 4 jumps of 1.

3. Draw and label a backwards jump of 10 on the number line. Land on 72.

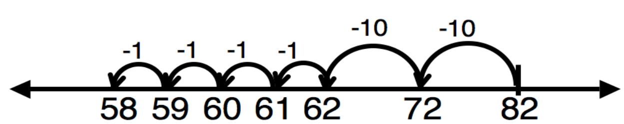
4. Draw and label 2nd backwards jump of 10 on the number line. Land on 62.



5. Draw and label four backwards jumps of 1 along the number line.

   We go back to 61, 60, 59, and finally **58.**

   Answer: 58



In the above example, we subtracted 82 - 24 by starting at 82 and subtracting 2 tens and 4 ones. There are many other correct ways to solve this problem on a number line. **As with addition, there is typically always more than one path to the solution, and any efficient and accurate path is acceptable.**

Turn to the following page to see the above problem solved two more ways using number lines.

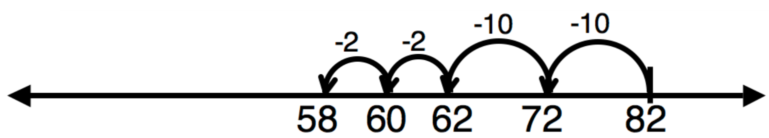
Number Lines to Count Back (continued)

Solve: 82 - 24

In Context: There were 82 students in the fourth grade. 24 of the students did a project for extra credit. How many students will not receive extra credit?

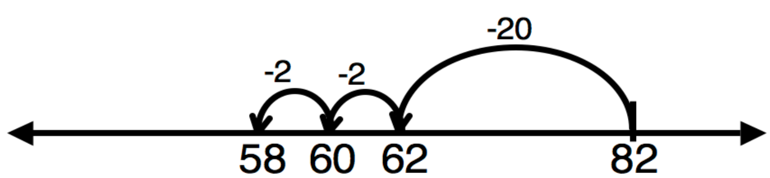
Here is the same problem, 82 - 24, solved two more correct ways:

On this number line, the student decomposed the 24 into 10, 10, 2, and 2. This takes advantage of friendly numbers. By splitting the 4 into 2 and 2, the movement backwards across 60 is easier.



82 - 24 = 58

On this number line, the student moved backwards by 20 in one jump. Then, the student used friendly numbers, splitting up the 4 into 2 and 2. The student subtracted 2 to land on 60, before going back 2 more to the answer of 58.



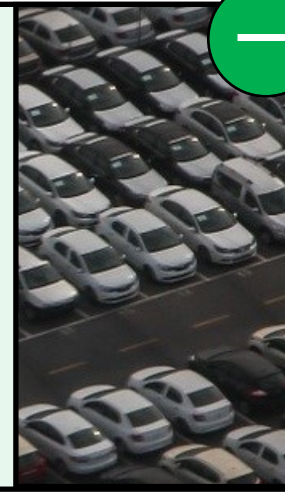
82 - 24 = 58

Tips:

- Encourage students who may be subtracting with only jumps of one to begin taking larger jumps as they show they are ready. As students gain experience and confidence, they can combine numbers with more efficient and clever combinations.
- As with addition, students can mentally decompose numbers before and during subtraction, in order to use a more strategic approach. Students can make steps easier by subtracting friendly numbers, such as 10 or 20. Students can also use jumps they design to land on round numbers, such as changing 43 - 6 into 43 - 3 - 3, so as to land neatly on 40 before going back further.

Number Lines to Count On

(Finding the Difference)



Not all subtraction involves "taking away from." Many subtraction problems involve finding a *difference* between two numbers: 85 - 83 = 2 *because 85 and 83 are 2 jumps apart*. Therefore, students can also solve subtraction problems on a number line by finding how many jumps are *between* the two numbers. To do this, students start at the smaller number (subtrahend) and draw and label jumps forward until the larger number (minuend) is reached. The total of the jumps will tell you the difference (your answer).

Solve: 125 - 79

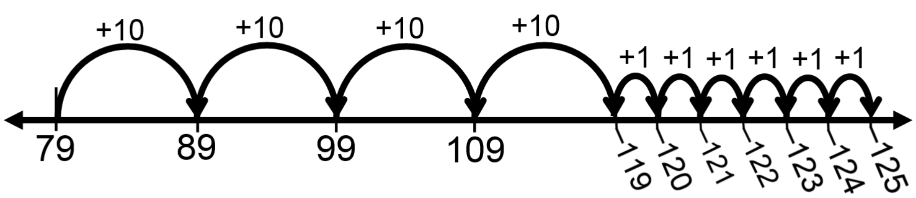
minuend subtrahend

In Context: There are now 125 new cars on the car lot. Before today's delivery arrived, there were only 79 cars on the lot. How many new cars were delivered today?

This is one way to solve this problem with a number line:

Steps:

1. Draw a number line and mark 79 as a starting point, leaving space to count on.
2. Jump ahead by tens until to get as close to the target of 125 as possible without going over.
3. Jump ahead by ones until you reach 125.

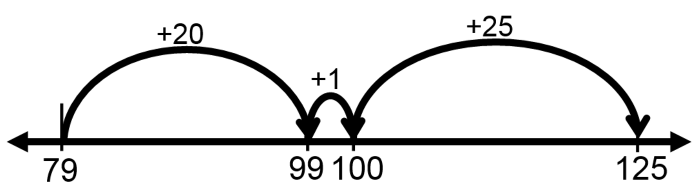


4. Add up the numbers that were added to get to 125.
   10 + 10 + 10 + 10 + 1 + 1 + 1 + 1 +1 + 1 = 46

The total difference between 79 and 125 is **46.** **Answer: 46**

During the delivery, 46 cars were added to the original 79 to reach the current total of 125.

The same problem solved with some larger and more efficient jumps:



This student added 20 and then 1 more, in order to land on 100. Then another jump of 25 was needed, going from 100 to 125. The total difference (20 + 1 + 25) between 79 and 125 is still **46.**

Answer: 46

Standard Algorithm Without Renaming

The standard algorithm is used when students have a deep understanding of place value. These examples have no renaming (some people call this "borrowing").

Solve: 68—23

In Context: In a pet store, there were 68 animals to be adopted. 23 families adopted pets over the weekend. How many animals were still waiting to be adopted?

Steps:

1. Stack the two numbers neatly above one another.
   Line up digits in the ones, tens, and hundreds
   under the corresponding place values.

$$\begin{array}{r} 68 \\ -\,23 \\ \hline \end{array}$$

Because the digits in the top number (minuend) are all larger than the corresponding digits in the bottom number (subtrahend), no renaming is needed to solve this problem.

2. Subtract the digits in the ones place. 8 - 3 = 5
3. Subtract the digits in the tens place. 6 - 2 = 4 (60 - 20 = 40)

$$\begin{array}{r} 68 \\ -\,23 \\ \hline 45 \end{array}$$

Answer: 45

Tips:

- Remind children of the values of the digits. For instance, after solving the problem above: You might ask your child questions such as:

  What is the value of the digit 4 in your answer? (40)
  How did we get 40? (subtracting 60– 20)

- Sometimes a student may incorrectly stack the numbers, such as in this example:

$$\begin{array}{l} \;\;243 \\ -\,21 \\ \hline \end{array}$$

This may reveal that a child does not yet have a clear and complete understanding of place values. If this occurs, it may be time to revisit expanded form, base-10 models, or other concrete representations.

Standard Algorithm With Renaming

The standard algorithm is used when students have a deep understanding of place value. While it does not show the true value of the digits, it can be an efficient strategy for students who have a solid background. Once renaming is involved, it becomes even more crucial that students know that ten ones make 10, ten tens make 100, etc.

Solve: 234 — 56

In Context: After the farm market closed, the Anderson family had 56 jars of their world famous jams and jellies remaining. They began the weekend with 234 jars. How many jars did they sell?

Steps:

1. Stack the two numbers above one another. Make sure the greater number is on the top. Line up digits in the ones, tens, and hundreds under the corresponding place values.
2. Start with the ones column. Subtract the bottom number from the top number. Can you subtract 4 - 6? No. There are not enough ones to take 6 away.

$$\begin{array}{r} 234 \\ -\ 56 \\ \hline \end{array}$$

3. Go to the tens column. Transfer 1 ten (worth 10 ones) from the tens column. Show this by marking out the 3 and showing that there are now only 2 tens left in the tens column. We are trading this ten for 10 ones.
4. Add the 10 ones that were transferred to the ones column: (10 + 4 = 14) You have 14 ones so you can take 8 away. Now subtract the ones: 14 - 6 = 8.
5. Look at the tens column. Subtract the bottom tens digit from the top tens digit if you can. Can you subtract 2 tens - 5 tens? No. There are not enough tens to take 5 tens away.

$$\begin{array}{r} \scriptstyle 2\ 14 \\ 2\cancel{3}\cancel{4} \\ -\ 56 \\ \hline 8 \end{array}$$

6. Go to the hundreds column. Transfer 1 hundred (worth 10 tens) from the hundreds column. Show this by marking out the 2 and showing that there is now only 1 left in the hundreds place.
7. Add the 10 tens that were transferred to the tens column. 10 + 2 = 12. Subtract the tens: 12 tens - 5 tens = 7 tens.

$$\begin{array}{r} \scriptstyle 1\ 12\ 14 \\ \cancel{2}\cancel{3}\cancel{4} \\ -\ 56 \\ \hline 78 \end{array}$$

8. Look at the hundreds column. Subtract the bottom hundreds digit from the top hundreds digit.
9. Subtract 1 hundred - 0 hundreds. 100 - 0 = 100.

$$\begin{array}{r} \scriptstyle 1\ 12\ 14 \\ \cancel{2}\cancel{3}\cancel{4} \\ -\ 56 \\ \hline 178 \end{array}$$

Answer: 178

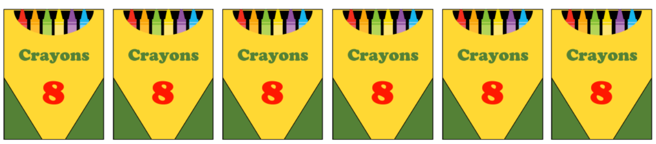
MULTIPLICATION INTRODUCTION

Multiplication is a way to quickly find the total amount in a collection of equal groups or sets. Let's take a quick look at the 3 most common situations that involve multiplication.

1. Equal Groups or Sets

In many multiplication problems, there are a given number of groups or sets, each of which is equal in size or quantity. Multiply the **number of groups × the number in each group** in order to find the total (product).

Derrick bought 6 packs of crayons. Each pack of crayons has 8 crayons inside. How many crayons did he buy altogether?



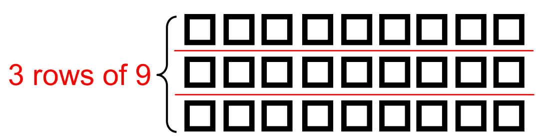
number of groups	×	number in each group	= total
6		8	= 48
number of boxes	×	crayons in each	total crayons

6 packs of 8 crayons make 48 crayons in all. 6 × 8 = **48**

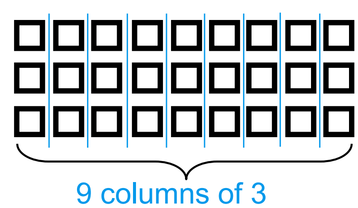
2. Arrays

Arrange items into equal rows and columns to form a rectangular array. Rather than counting all of the items in the array, you can multiply the **number of rows × the number of columns,** in order to find the total (product).

The chairs for the meeting were arranged in 3 rows of 9. How many chairs were set up?



Or you could look at it as:

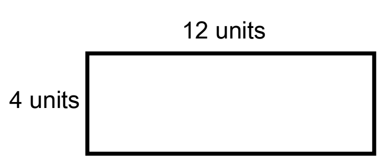


27 chairs were set up. 3 × 9 = 27 or 9 x 3 = 27

3. Area (of a rectangle):

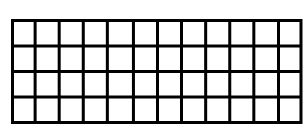
Area of a rectangle is calculated by multiplying length to width to find the total squares within the rows and columns.

Find the area of a rectangle, if the length is 12 and the width is 4.



→

Inside the shape are 4 rows of 12 squares.



4 x 12 = 48 The area is 48 square units.

Skip Counting to Multiply

Skip counting means counting by the same number repeatedly.
Example: Skip counting by three: 3, 6, 9, 12, 15, 18, 21…
This method is used to solve basic multiplication problems.
Students can skip count the correct number of jumps to find the total.
The total in multiplication is called the **product**.

Solve: 4 X 5

In Context: At Freedom Elementary School, there are 4 classrooms in the second grade. Each classroom has 5 class pets. How many pets are in the second grade classes?

Steps:

1. Think about the problem:

   Classroom #1 has 5 pets   Classroom #2 has 5 pets

   Classroom #3 has 5 pets   Classroom #4 has 5 pets

2. Skip count by fives (4 jumps) to find the product (total).

5,	10,	15,	**20**
(1x5)	(2x5)	(3x5)	(4x5)

Answer: There are **20** pets altogether.

Here is one more example:

6 friends each eat 3 cookies. How many cookies were eaten? **Solve:** 6 x 3

Steps:

1. Think about the problem:

   Friend #1 has 3 cookies   Friend #2 has 3 cookies   Friend #3 has 3 cookies

   Friend #4 has 3 cookies   Friend #5 has 3 cookies   Friend #6 has 3 cookies

2. Skip count by threes (6 jumps) to find the product (total).

3,	6,	9,	12,	15,	**18**
(1x3)	(2x3)	(3x3)	(4x3)	(5X3)	(6x3)

Answer: **18** cookies were eaten altogether.

Note: While early learners of multiplication may not recognize this at first, this problem could also be solved in fewer jumps by counting by 6, three times.

6, 12, **18**.   (3 x 6 = 6 x 3)

Tip:

This method typically only applies to solving basic multiplication math facts and patterns.

Repeated Addition to Multiply

As students are learning the concept of multiplication, they must understand that multiplication is a way to combine equal groups or sets. Repeated addition is just as it sounds: adding the same number repeatedly in order to find the total.

Solve: 17 X 4

In Context: A group of students filled 4 bags of apples at the apple orchard. Each bag contained 17 apples. How many apples did the students pick?

Steps:

1. Think about the problem:

   Bag #1 contained 17 apples   Bag #2 contained 17 apples
   Bag #3 contained 17 apples   Bag #4 contained 17 apples

2. Write out the multiplication problem as repeated addition.

$$17 \times 4 = 17 + 17 + 17 + 17$$

3. Use any addition strategies to solve the problem. Here are three different methods:

$$17 + 17 = 34$$
$$34 + 17 = 51$$
$$51 + 17 = \mathbf{68}$$

$$\begin{array}{r} {}^{2} \\ 17 \\ 17 \\ 17 \\ +17 \\ \hline \mathbf{68} \end{array}$$

$$\underbrace{17 + 17}_{34} + \underbrace{17 + 17}_{34} =$$
$$\underbrace{34 + 34}_{\mathbf{68}} =$$

The students collected 68 apples.

Tips:

- Third graders learn about the commutative property of multiplication. Repeated addition is another way for students to learn the importance of the commutative property of multiplication, which shows that 4 x 13 is the same as 13 x 4.

  $4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 = 52$ or
  $13 + 13 + 13 + 13 = 52$

- Repeated addition helps illustrate the meaning of the multiplication operation. Understanding the repeated addition concept will also have value for students **IN LATER GRADES** as they first begin to learn how to multiply fractions and decimals by whole numbers **in grades 4-5**.

  Grade 4 Application: 3 x ¼ = ¼ + ¼ + ¼ = ¾

  Grade 5 Application: 5 x 0.8 = 0.8 + 0.8 + 0.8 + 0.8 + 0.8 = 4.0

Using Models of Sets to Multiply

In this early and important multiplication method, students build or draw a model of equal sets to represent and solve multiplication problems. Sets must be equal in size.
This method can be used to solve basic multiplication problems.
Students count the total number of objects in the sets to find the **product**.

Solve: 4 X 6

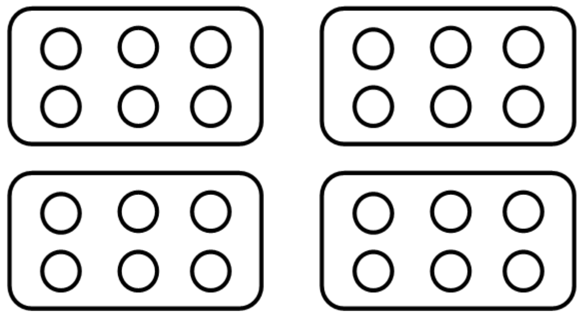
In Context: Tom baked mini pizzas for the annual football party. He filled 4 trays with pizzas. Each tray held 6 pizzas. How many mini pizzas did Tom bake?

Steps:

1. Draw four trays (symbols are fine).
2. Draw 6 pizzas in each tray.
3. Add or count to find the total number of pizzas.

   6 + 6 + 6 + 6 = **24**

   Answer: **24**



Another example:

Solve: 5 X 7

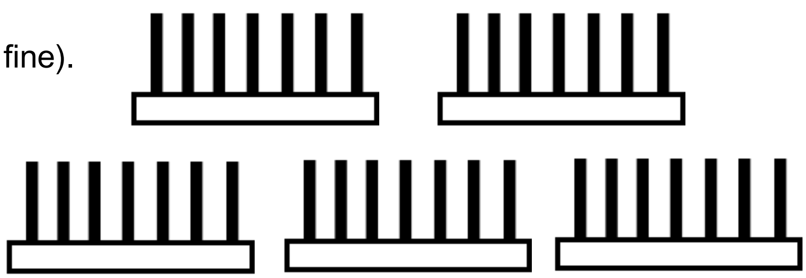
In Context: There are 5 bookshelves with 7 books on each shelf. How many books are there altogether?

Steps:

1. Draw five shelves (symbols are fine).
2. Draw 7 books in/on each shelf.
3. Add or count to find the total number of books.

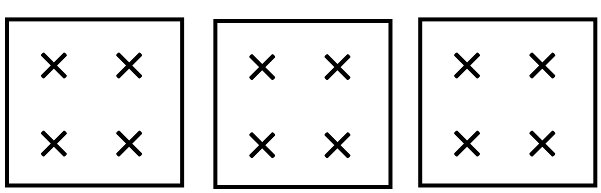
   7 + 7 + 7 + 7 + 7 = **35**

   Answer: **35**



Tips:

- Sets of everyday objects can be used to model multiplication. There is no need to rely solely on drawings. Make equal groups of pennies, toys, food...or anything around the house!
- Students may use skip counting to total the objects in the groups.
- When creating drawings, students should keep them very simple. Students can represent 3 barns with 4 horses in each barn by drawing rectangles for the barns and x's for the horses.



Arrays for Multiplication

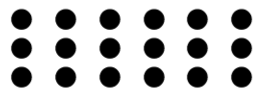
An array is a visual organization of symbols, shapes or objects in equal rows and columns. It is typically rectangular. This is a critical model for multiplication, in which:

the number of rows × the number of columns = the total objects

Students draw arrays and then use skip counting, addition, or other strategies to total up the objects in the arrays.

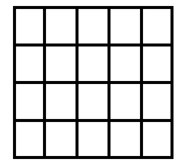
Arrays can be drawn with symbols such as **x's, dots, or a grid**:

2 × 5 or 5 × 2



3 × 6 or 6 × 3

5 × 4 or 4 × 5



Solve: 6 X 8

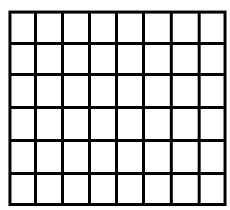
In Context: There are 6 tables at a birthday party. Each table has 8 chairs. How many guests can have a seat at the party?

Steps:

1. Draw an array, or use objects such as square tiles, to create a 6 x 8 array such as:

8 columns

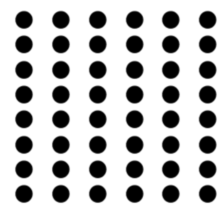
6 rows



or

6 columns

8 rows



2. Add, skip count, count, or use other methods to find the total. Here are several ways:

6, 12, 18, 24, 30, 36, 42, **48**

8, 16, 24, 32, 40, **48**

6 + 6 + 6 + 6 + 6 + 6 + 6 + 6 = **48**

8 + 8 + 8 + 8 + 8 + 8 = **48**

Tips:

- Students can model arrays with a variety of household items such as cereal or pennies.
- Point out arrays that you observe in your everyday life, like cartons of eggs, and talk with your child about them and the multiplication that they represent.
- When creating arrays to fit story problems, students should keep them very simple. Students can create arrays with simple dots, circles, boxes, or x's to represent anything.
- Graph paper is a terrific tool for modeling larger problems such as 25 x 35.

Multiplying Using Number Lines

Students can use the same number line drawings they used for addition to represent multiplication as repeated addition of equal groups. Students draw a number line and represent the equal groups or sets as equal jumps along the line, starting at 0. Once all the sets or groups have been added, the stopping point is the total.

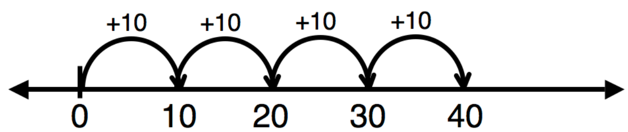
Solve: 4 ×10

In Context: Mark had 4 presents to be wrapped. Each present required 10 inches of ribbon. How much ribbon will Tom need?

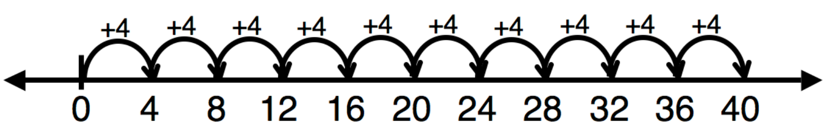
Steps:

1. Draw a blank number line and mark 0 as a starting point.
2. 4 x 10 can represent four tens or ten fours, and can be shown either way on a number line. In this example, it is easier and faster to count by tens. Draw 4 jumps of 10 forward on the number line: 0, 10, 20, 30, 40.

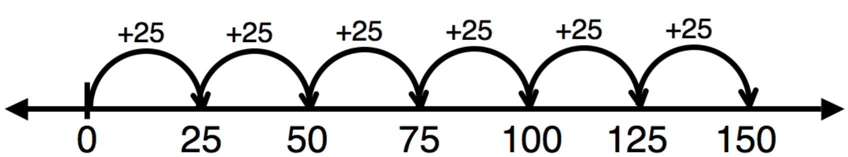
   Answer: 40



Here is the same example using 10 groups of 4. 4 ×10 = 10 × 4



This is a different example using a number line to calculate 6 x 25, shown as 6 jumps of 25.



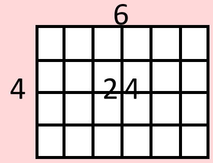
Students will be able to explore which representation will be most efficient. Imagine what 25 jumps of 6 would look like!

Tip: This strategy is not an efficient long-term strategy, but it helps build students' understanding of the multiplication concept, and it connects multiplication to students' understanding of addition.

Area Model of Multiplication

The area model for multiplication is based on the concept of **area of a rectangle**. Area of a rectangle is calculated by multiplying length times width to find the total squares within the rows and columns. The total square units inside the rectangle are the product.

Example: 4 x 6 = 24 square units

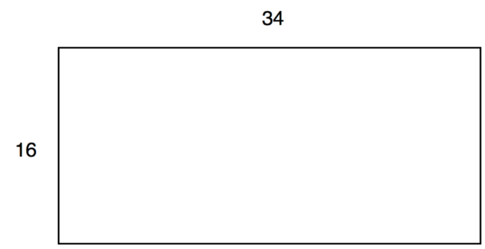


Solve: 34 X 16

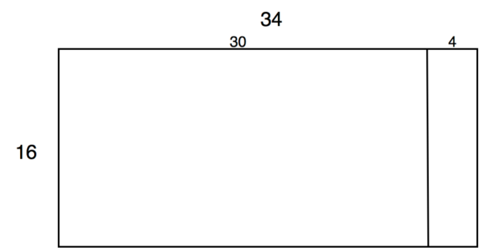
In Context: In a hot dog eating contest, the contestants ate 34 large packs of hot dogs, with 16 hot dogs in each pack. How many hot dogs did the contestants eat in all?

Steps:

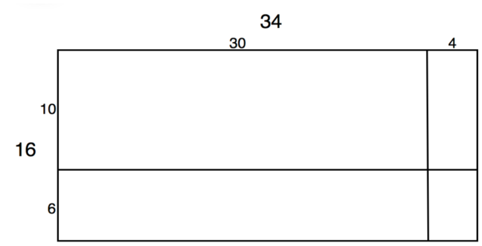
1. Draw a rectangle with the length (34) and width (16) labeled as the two factors to be multiplied.



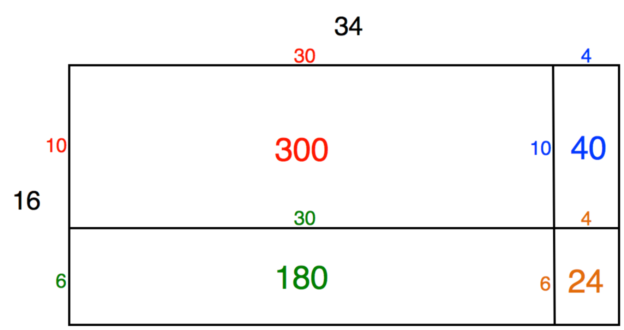
2. Decompose/break apart 34 into friendlier numbers that can be multiplied easily in your head. Students often decompose numbers into the values of each digit, so we will just split 34 into 30 and 4. Write the 30 and 4 on the top of the area model. Draw a vertical line inside the rectangle, and try to show that 30 is larger than 4.



3. Decompose/break apart 16 into friendlier numbers that can easily be multiplied in your head. We will split 16 into 10 and 6. Draw a horizontal line inside the rectangle, and try to show that 10 is larger than 6.

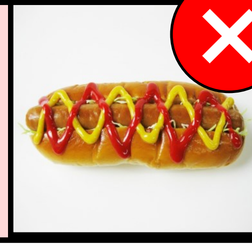


4. Use mental math to multiply length and width and find the area of each subsection.
   - Multiply 30 × 10 and write the product, 300, inside the upper left hand section.
   - Multiply 30 × 6 and write the product, 180, inside the lower left hand section.
   - Multiply 4 ×10 and write the product, 40, inside the upper right hand section.
   - Multiply 4 × 6 and write the product, 24, inside the lower right hand section.



5. Add up the 4 smaller areas to find the total: 300 + 180 + 40 + 24 = 544 **Answer: 544**

Area Model for Multiplication (Continued)



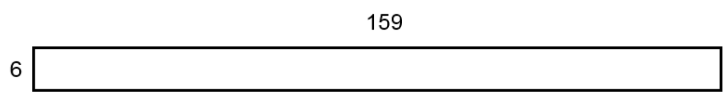
The area model for multiplication is based on the concept of **area of a rectangle**. Area of a rectangle is calculated by multiplying length to width to find the total squares within the rows and columns. The total square units inside the rectangle are the product.

An example with a 3-digit number multiplied by a 1-digit number:

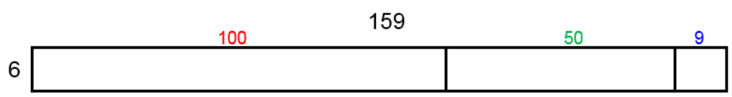
Solve: 159 × 6

Steps:

1. Draw a rectangle with the length (159) and width (6) labeled as the two factors to be multiplied.

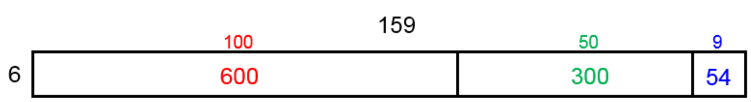


2. Break apart 159 into friendlier numbers that can be multiplied easily in your head: 100 + 50 + 9. Write the 100, 50, and 9 on the top of the area model. Draw vertical lines inside the rectangle to separate the 3 sections. Write the other factor, 6, on the side as the width.



3. Mentally multiply length x width to find the area of each section.

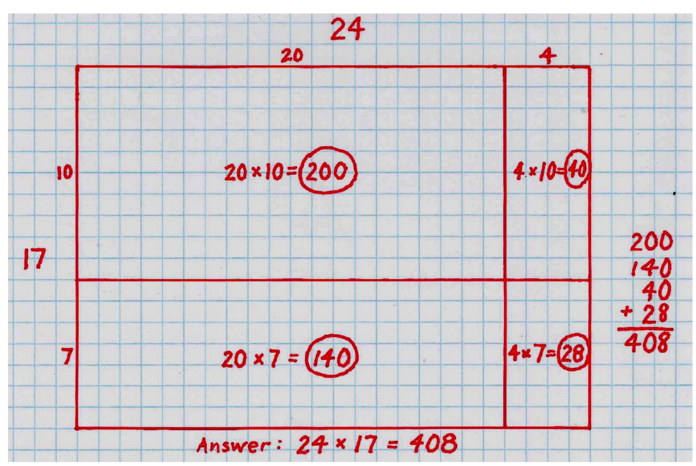
- Multiply 100 × 6 and write the product, 600, inside the 1st section.
- Multiply 50 × 6 and write the product, 300, inside the 2nd section.
- Multiply 9 × 6 and write the product, 54, inside the 3rd section.



4. Add up the 3 smaller areas to find the total: 600 + 300 + 54 = 954 **Answer: 954**

Tip:

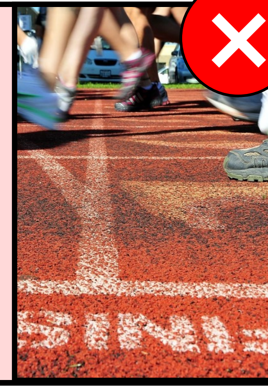
Graph paper is a great tool for creating area models. It is particularly useful when students are first learning about the area model, or when multiplying larger numbers. Trace around the area you want, in order to make a quick model that actually shows the square units. Students can also trace and label smaller sections of the figure and use partial products to solve a large area on graph paper, adding the areas of the pieces together to find the total area of the figure.



Partial Products (Using Expanded Form)

Partial products is a strategy where students use **expanded form** to break larger numbers into smaller, round numbers that are easier to mentally multiply together. Then students add the partial products together to find the total.

The **expanded form** of a number is the value of each digit written out separately in an addition expression. Example: The expanded form of 834 is ***800 + 30 + 4***.



Solve: 35 X 14

In Context: There were 14 girls on the track team, who each set a goal to run 35 laps around the track in fifty minutes. How many laps would they run in all?

35 = 30 + 5
14 = 10 + 4

Steps:

1. Decompose/break apart 35 into 30 + 5
2. Decompose/break apart 14 into 10 + 4.

$$\begin{array}{r} 35 \\ \times\, 14 \\ \hline \end{array}$$

3. Multiply each value from 35 (30 and 5) by each value from 14 (10 and 4).

- Multiply 4 (from 14) X 5 (from 35). This equals 20.
- Multiply 4 (from 14) X 30 (from 35). This equals 120.
- Multiply 10 (from 14) X 5 (from 35). This equals 50.
- Multiply 10 (from 14) X 30 (from 35). This equals 300.

$$\begin{array}{r|l} 35 & \\ \times\, 14 & \\ \hline 20 & (4 \times 5) \\ 120 & (4 \times 30) \\ 50 & (10 \times 5) \\ 300 & (10 \times 30) \end{array}$$

4. Add all of the products together. 300 + 50 + 120 + 20 = **490**

Answer: 490

$$\begin{array}{r|l} 35 & \\ \times\, 14 & \\ \hline 20 & (4 \times 5) \\ 120 & (4 \times 30) \\ 50 & (10 \times 5) \\ +\, 300 & (10 \times 30) \\ \hline 490 & \end{array}$$

Tips:

- Students can organize the steps however they like, **but** most students stack the numbers. Neat stacking makes the addition of the partial products easier.
- Each digit from the bottom number must be multiplied by each digit in the top number. ***However, it may be wise to do the steps in the same order as you would in the traditional standard algorithm to avoid confusion later on (as shown above).***

Standard Algorithm Without Regrouping

The standard algorithm is used when students have a deep understanding of place value, multiplication, and how the digits represent equal groups. This strategy is often confusing for students who don't yet have these concepts.

Solve: 412 X 3

In Context: The farmer's new front fence will be 412 yards long. A yard is equal to 3 feet. How many feet long will the front fence be?

Steps:

1. Stack the two numbers neatly above one another. Take care to line up digits in the ones, tens, and hundreds under the corresponding place values.

$$\begin{array}{r} 412 \\ \times\ \ 3 \\ \hline \end{array}$$

Multiply 3 x 412 by multiplying 3 times each digit in the top number.

2. Multiply the digit 3 in the ones place in the bottom number by the digit 2 in the ones place in the top number. 3 X 2 = 6 (ones)
3. Write the product, 6, in the ones place underneath the ones column.

$$\begin{array}{r} 412 \\ \times\ \ 3 \\ \hline 6 \end{array}$$

4. Multiply the digit 3 in the ones place in the bottom number by the digit 1 in the tens place in the top number. 3 X 1 = 3 (tens)
5. Write the product, 3, in the tens place.

$$\begin{array}{r} 412 \\ \times\ \ 3 \\ \hline 36 \end{array}$$

6. Multiply the digit 3 in the ones place in the bottom number by the digit 4 in the hundreds place in the top number. 3 X 4 = 12 (hundreds)
7. Write the product, 12, starting in the hundreds place. Because you have 12 hundred, the answer spills over into the thousands place. So, the 2 hundreds go beneath the hundreds column, and the 1 thousand goes to the left.

$$\begin{array}{r} 412 \\ \times\ \ 3 \\ \hline 1{,}236 \end{array}$$

Answer: 1,236

Tip: If your child becomes confused by the steps, try solving the problem using the partial products method on the previous page. Then retry the problem with the standard algorithm, making connections between the steps.

$$\begin{array}{r|l} 412 & \\ \times\ \ 3 & \\ \hline 6 & (3 \times 2) \\ 30 & (3 \times 10) \\ 1200 & (3 \times 400) \\ \hline 1{,}236 & \end{array}$$

$$\begin{array}{r} 412 \\ \times\ \ 3 \\ \hline 1{,}236 \end{array}$$

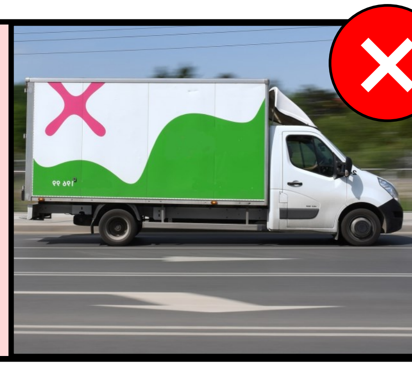
3 x 2 = 6

3 x 10 = 30

3 x 400 = 1200

Standard Algorithm With Regrouping

The standard algorithm is used when students have a deep understanding of place value, multiplication, and how the digits represent equal groups. While it does not show the true value of the digits, it is an efficient strategy for students who have a solid background.



Solve: 174 X 5

In Context: The delivery truck is loaded up with 5 new refrigerators. These lightweight refrigerators each weigh 174 pounds. What is the total weight of the cargo?

Steps:

1. Stack the two numbers neatly above one another. Line up digits in the ones, tens, and hundreds under the corresponding place values.

$$\begin{array}{r} 174 \\ \times\ \ 5 \\ \hline \end{array}$$

Multiply 5 x 174 by multiplying 5 by each digit in the top number.

2. Multiply the digit 5 in the ones place in the bottom number by the digit 4 in the ones place in the top number. 5 X 4 = 20 ones
3. Don't write 20 in the ones place below the line. Regroup the 20 ones to make 2 tens and 0 ones. Write the 2 above the tens column (over the 7 as shown in red). Write the remaining 0 ones below the line in the ones place.

$$\begin{array}{r} 2 \\ 174 \\ \times\ \ 5 \\ \hline 0 \end{array}$$

4. Multiply the digit 5 in the ones place in the bottom number by the digit 7 in the tens place in the top number. 5 X 7 = 35 tens
   Add the **2** extra tens on top from the previous step. 35 + 2 = 37 tens
5. You cannot write 37 in the tens place below the line. Regroup the 37 tens (370) to make 3 hundreds and 7 tens. Write the 3 above the hundreds column (over the 1 as shown in blue). Write the remaining 7 tens below the line in the tens place.

$$\begin{array}{r} 32 \\ 174 \\ \times\ \ 5 \\ \hline 70 \end{array}$$

6. Multiply the digit 5 in the ones place in the bottom number by the digit 1 in the hundreds place in the top number. 5 X 1 = 5 hundreds
   Add the **3** extra hundreds on top from the previous step. 5 + 3 = 8 hundreds
7. Write the 8 hundreds in the hundreds place.

$$\begin{array}{r} 32 \\ 174 \\ \times\ \ 5 \\ \hline 870 \end{array}$$

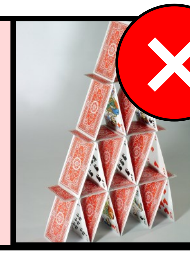
Answer: 870

Tip:

Remember to only regroup to the top of the next place value when there is at least ten in any given place value at the end of a step. Some problems may involve regrouping only the tens, or only the hundreds, or neither.

Standard Algorithm With 2-Digit Numbers

The standard algorithm is used when students have a deep understanding of place value, multiplication and how the digits represent equal groups. While it does not show the true value of the digits, it can be an efficient strategy for students who have a solid background.



Solve: 52 x 48

In Context: José and Marcus are attempting to build a record size house of cards. They hope to use 48 decks of cards. Each standard deck contains 52 cards. How many cards will be in the card house if they are successful?

Steps:

1. Stack the two numbers neatly above one another. Line up digits in the ones, tens, and hundreds under the corresponding place values.

$$\begin{array}{r} 52 \\ \times\, 48 \\ \hline \end{array}$$

First, multiply 8 decks of cards x 52 cards in each.

2. Multiply the digit 8 in the ones place in the bottom number by the digit 2 in the ones place in the top number. 8 X 2 = 16 ones
3. Don't write 16 in the ones place below the line. Regroup the 16 ones to make 1 ten and 6 ones. Write the 1 above the tens column (over the 5 as shown in red). Write the remaining 6 ones below the line in the ones place.

$$\begin{array}{r} {}^{1} \\ 52 \\ \times\, 48 \\ \hline 6 \end{array}$$

4. Multiply the digit 8 in the ones place in the bottom number by the digit 5 in the tens place in the top number. 8 X 5 = 40 tens
   Add the 1 extra ten on top from the previous step. 40 + 1 = 41 tens

Tip: Cross out the 1 after you add it, so you don't accidentally add it again in the next step.

5. Write the 41, starting in the tens place. Because you have 41 tens, this spills over into the hundreds place.

$$\begin{array}{r} {}^{1} \\ 52 \\ \times\, 48 \\ \hline 416 \end{array}$$

Next, multiply the other 40 decks of cards x 52 cards in each.

6. Multiply 40 by 52. Start by placing a 0 underneath the 6 in the ones place of the answer section beneath the line. This 0 is called a placeholder. It represents the 0 in 40, because you are multiplying by 52 by 40, not just 4.
7. Multiply the digit 4 in the tens place in the bottom number by the digit 2 in the ones place in the top number. 4 X 2 = 8 tens.
8. Write the 8 in the tens place next to the placeholder 0.

$$\begin{array}{r} \cancel{1} \\ 52 \\ \times\, 48 \\ \hline 416 \\ 80 \end{array}$$

9. Multiply the digit 4 in the tens place in the bottom number by the digit 5 in the tens place in the top number. 4 X 5 = 20 hundreds. *Don't add the old 1 on top!*
10. Write the 20, starting in the hundreds place. Because you have 20 hundreds, this spills over into the thousands place.
11. Add the 416 (from 8 x 52) plus the 2,080 (from 40 x 52).
    416 + 2080 = 2,496. **Answer: 48 x 52 = 2,496**

$$\begin{array}{rr} & \cancel{1} \\ & 52 \\ & \times\, 48 \\ \hline (8 \times 52) \longrightarrow & 416 \\ + (40 \times 52) \longrightarrow & +2080 \\ \hline (48 \times 52) \longrightarrow & 2{,}496 \end{array}$$

DIVISION INTRODUCTION

Division situations usually involve a known total that is divided into equal groups.
The total is equal to the number of groups × the number in each group.

Total

divided into:

Number of Groups × Number in Each Group

Students are often dividing for one of two reasons:

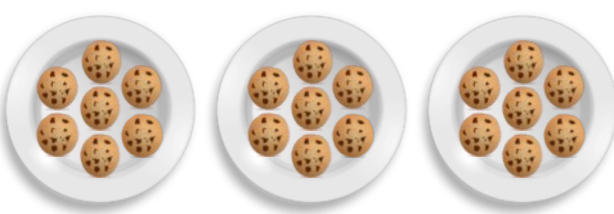
A. In some problems, the number of equal groups is given, but we need to figure out how many items will be in each group.

Total

divided into:

Number of Groups × Number in Each Group ?

We have 21 cookies divided onto 3 plates. How many cookies will go on each plate?



7 cookies <u>on each plate</u> is 21 cookies.

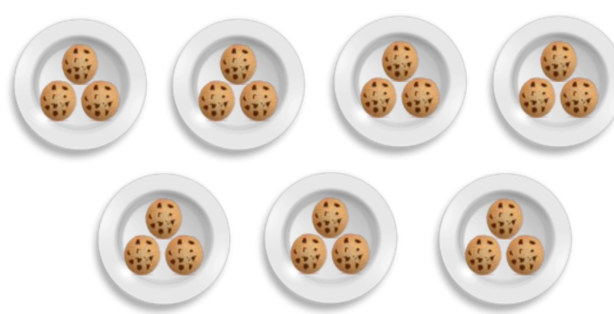
B. In some problems, the size of the equal groups is given, but we need to figure out how many groups we can make.

Total

divided into:

Number of Groups ? × Number in Each Group

We have 21 cookies divided onto plates with 3 cookies on each plate. How many plates?



7 <u>plates</u> of 3 cookies make 21 cookies.

Using Models of Sets to Divide

In this early division model, students build or draw models of equal sets to represent and solve division problems. They may draw equal groups and distribute all of the items into each group, or make all the groups of a given size, until they reach the total. The answer is called the **quotient.**

Example #1: **Solve:** 18 ÷ 3

In Context: There are 18 students in a science classroom. They are put into three equal groups for a class project. How many students will be in each group?

Steps:

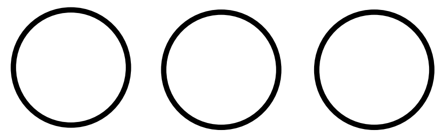
1. The problem states that there are 3 groups, and we need to find <u>how many are in each group</u>.

Total

divided into:

Number of Groups × Number in Each Group ?

2. To solve this problem, start by drawing 3 shapes, such as circles or boxes, to represent each of the 3 groups.



3. Add one student into each group. Represent 18 students with stars or other symbols until all 18 students are placed in groups.
4. Count the number of students (stars) in each group.
The quotient (answer) is 6. There are 6 students in each group.

Example #2: **Solve:** 40 ÷ 5

In Context: Turtletown Elementary needs to order spiral notebooks for all 40 of its 3rd graders. The notebooks come in packs of 5. How many packs of notebooks will they need to order so that all 40 students receive one?

Steps:

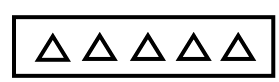
1. This problem states that there are equal groups of 5. We need to find <u>how many groups of 5 make 40</u>.

Total

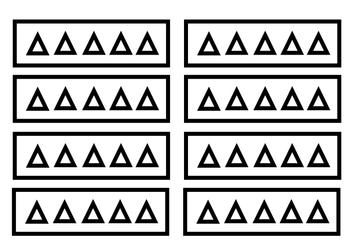
divided into:

Number of Groups ? × Number in Each Group

2. Draw a quick shape, such as a circle or box, to represent a pack of 5.



3. Draw another pack, and then another, keeping track of how many total notebooks are included, until there are 40 total notebooks.
4. Count how many packs were made. **The quotient (answer) is 8.**
It takes 8 packs of 5 to make a total of 40 notebooks.



Using Models of Sets to Divide (Continued with remainders)

Some problems have numbers that may not divide evenly, such as 11 divided by 5. After forming equal groups, a partial group of extra items remains. The number of leftover items that cannot form a complete group or cannot be shared with each existing group is called the **remainder**.

Example #3 with a remainder: **Solve:** 20 ÷ 3

In Context: There are 20 students in a science classroom. They are put into three equal groups for a class project. How many students will be in each group?

Steps:

1. The problem states that there are three groups, but we want to find out <u>how many are in each group.</u>
2. Draw some quick shapes, such as a circles or boxes, to represent each of the 3 groups.
3. Add one student into each group (students are represented with stars) until all 20 are placed.

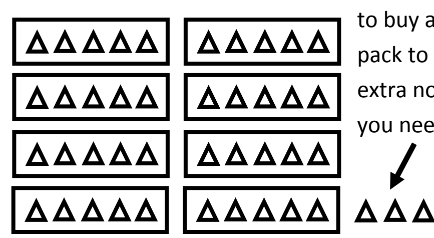
4. Count the number of students (stars) in each group.
   The number 20 does not evenly split into three groups, so you can put 6 students in each group, but this problem has a **remainder** of 2 extra students.
5. **The quotient (answer) is 6 with a remainder of 2**. There are 6 students in each of the 3 groups, with two students left over. **(Students may realize that two of the groups would have to have 7 students.)**

Example #4 with a remainder: **Solve:** 43 ÷ 5

In Context: Turtletown Elementary needs to order spiral notebooks for all 43 of its 3rd graders. The notebooks come in packs of 5. How many packs of notebooks will they need to order?

Steps:

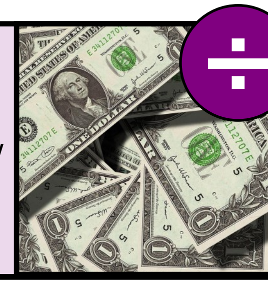
1. This problem states that there are equal groups of 5, but we want to find out <u>how many groups of 5 will make 43.</u>
2. Draw quick shapes, such as boxes, to stand for the packs of 5. Draw another pack, and then another, until you reach 43 total notebooks.
3. It takes 8 packs of 5 to make a total of 40 notebooks, but 9 packs would be 45. It is impossible to make exactly 43 using packs of 5.



The answer to 43 ÷ 5 is <u>8 with a remainder of 3</u>. **The answer to the *question* of how many packs to order is that the school would need to order <u>9 packs</u>, so everyone gets a folder.**

Using Number Lines to Divide

This strategy is used with students who are first learning to divide. Students can draw number lines and represent the equal groups or sets as equal jumps along the line. This strategy provides students with a visual model of division.

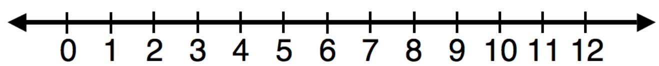


Example #1 **Solve:** $12 ÷ 4

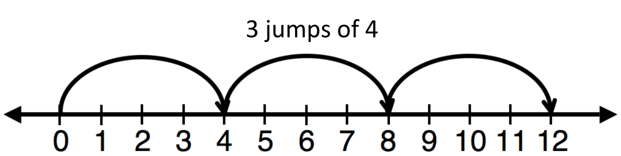
In Context: John has $12. He wants to buy some books that each cost $4. How many books can he buy?

Steps:

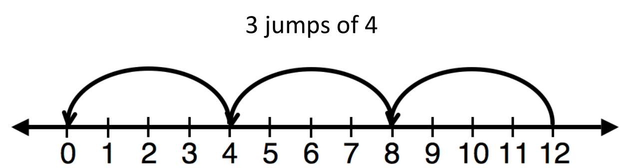
1. Draw a number line and label it with marks for every number from 0 to the total, which is 12.



2. Draw as many jumps of 4 as possible, moving either forward from 0 to 12, or backwards from 12 to 0.



or



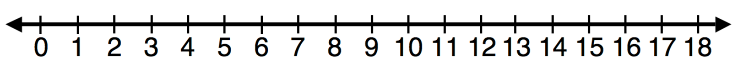
3. Count the number of jumps made, which tells how many groups of 4 are in 12.

There are 3 jumps of 4 to make 12. **The quotient (answer) is 3.** **12 ÷ 4 = 3**

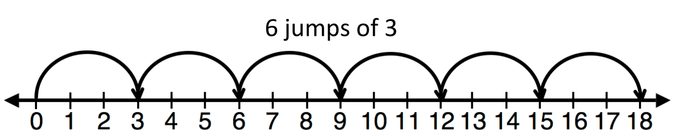
Example #2 **Solve:** 18 ÷ 3

Steps:

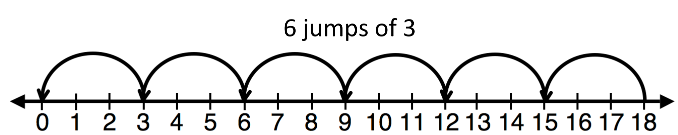
1. Draw a number line and label it with marks for every number from 0 to the total of 18.



2. Draw as many jumps of 3 as possible moving either forward from 0 to 18, or backwards from 18 to 0.



or



3. Count the number of jumps made, which tells how many groups of 3 are in 18.

There are 6 jumps of 3 to make 18. **The answer (quotient) is 6.** **18 ÷ 3 = 6**

Tips:

- This is a learning tool for students beginning to divide, rather than a long-term efficient strategy.
- This model is difficult to use with a story problem where you want to make a given number of groups, and need to find out how many are in each group.

Arrays for Division

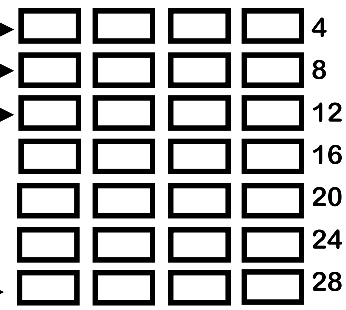
Because multiplication and division are opposite operations, the array model used for multiplication can also be used to divide. Arrays are rectangular arrangements of symbols, shapes or objects in equal rows and columns. In an array, ***the number of rows × the number of columns = the total objects.*** Students can create arrays to determine the number of groups or number in each group within a total.

Example #1: **Solve:** 28 ÷ 4

In Context: Angela is putting photos into a photo album. She has 28 photos, and the sheets in the album each hold 4 photos. How many sheets will she fill?

Steps:
Draw an array with equal rows of 4 to see how many groups of 4 it takes to make 28.

1. Draw 4 rectangles in the 1st row to stand for the 4 photos on sheet 1.
2. Draw 4 rectangles in the 2nd row to stand for the 4 photos on sheet 2.
3. Draw 4 rectangles in the 3rd row to stand for the 4 photos on sheet 3.
4. Keep drawing rows of 4 until you reach the total of 28 photos.



5. Count how many rows of 4 it took to make 28.

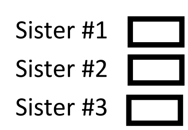
Answer: 7 rows of 4 make 28. 28 ÷ 4 = 7 Angela will fill 7 pages.

Example #2: **Solve:** 15 ÷ 3

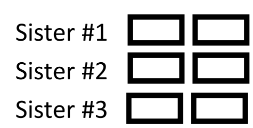
In Context: Three sisters work together to earn some money raking leaves. They earned $15. How many dollars go to each sister?

Steps:
Draw an array with 3 equal rows to see how many dollars each sister will get (in each row).

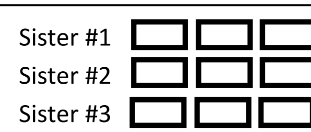
1. Draw 1 rectangle (dollar bill) in each of the 3 rows to give each sister $1.



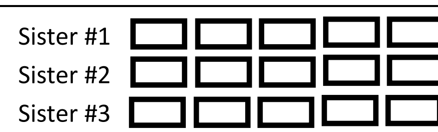
2. Add a 2nd rectangle in each row to give each sister another $1.



3. Add a 3rd rectangle in each row to give each sister another $1.



4. Keep giving each sister a dollar until all $15 have been handed out.



5. Count how many rectangles (dollars) are in each row.

Each row has 5 dollars. Answer: 15 ÷ 3 = 5 Each sister will get $5.

Arrays for Division
(Continued with Remainders)

Some problems have numbers that may not divide evenly, such as 11 divided by 5. This situation leaves you with a partial group of extra items. In an array, extra items leave you with an unfinished row or column. The number of leftover items that cannot form a complete group or cannot be shared with each existing group is called the **remainder**.

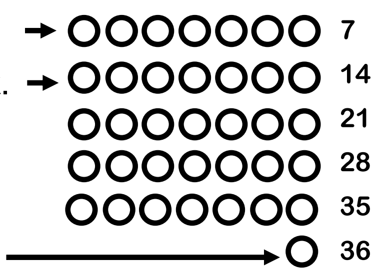
Example #3: Solve: 36 ÷ 7

In Context: Ronnie's birthday party is in 36 days. He wants to know how many weeks until his birthday, so his dad tells him to divide 36 by 7 days in each week. How many weeks make 36 days?

Steps:

Draw an array with equal rows of 7 to see how many sevens it takes to make a total of 36.

1. Draw 7 circles in the 1st row to stand for the 7 days in the 1st week.
2. Draw 7 circles in the 2nd row to stand for the 7 days in the 2nd week.
3. Keep drawing rows of 7 days until you reach the total of 36 days.



5 rows make 35 days, so you need 1 extra day to make 36 days.

4. Count how many rows of 7 it took to make 35.

   There are 5 rows with 1 extra circle. This means there is a remainder of 1.

Answer: There are 5 rows (weeks) and a remainder of 1 extra day.

36 ÷ 7 = 5 with a remainder of 1.   Ronnie's party will be in 5 weeks and 1 day.

Tips:

- It is great to use everyday objects to form arrays. Divide up everyday household items into equal rows and columns to answer simple division questions.
- Students can create arrays with simple dots, circles, boxes, or x's to represent anything.
- Graph paper is a terrific tool for modeling larger problems such as 96 divided into 6 equal rows.
- Remember that you can orient a rectangular array either vertically or horizontally.
- This is not an efficient model for long term use. It is a learning tool for students beginning to divide.

Partial Quotients

Partial quotients let students divide the total in manageable chunks. Students can multiply by larger numbers and solve the problem quickly in a few steps, or take more steps using smaller numbers, based on their confidence and proficiency level.

Solve: 152 ÷ 4

In Context: Parents made 152 cookies for a bake sale at school. They are to be packed into bags that are each filled with 4 cookies. How many bags of cookies will be filled for the parents to sell?

In each step using partial quotients, multiply to make as many groups of 4 as possible, without going over 152. After multiplying, subtract the amount of cookies already grouped into bags. Keep making groups and subtracting them until you run out of cookies. Then add the number of groups (bags) of cookies you have made to find the quotient (ans.

Step 1

Set up the problem 152 ÷ 4 this way.

Divisor	Work	Groups
4	1 5 2	
		4 × 10

Try starting with 10 groups of 4 by multiplying by 10.

4 x 10 = 40

10 bags hold 40 cookies.

Step 2

Divisor	Work	Groups
4	1 5 2	
	- 4 0	4 × 10
	1 1 2	

We started with 152 cookies, so we subtract the 40 cookies we just bagged.

112 cookies are left.

Step 3

Divisor	Work	Groups
4	1 5 2	
	- 4 0	4 × 10
	1 1 2	
	- 8 0	4 × 20
	3 2	

A. Now try doubling the 10 and multi-ply by 20 this time.

4 x 20 = 80

20 bags hold 80 cookies.

B. Subtract another 80 cookies.

Only 32 are left.

Step 4

Divisor	Work	Groups
4	1 5 2	
	- 4 0	4 × 10
	1 1 2	
	- 8 0	4 × 20
	3 2	
	- 2 0	4 × 5
	1 2	

A. With just 32 cookies left, we cannot multiply by 10.

Let's try 5.

4 x 5 = 20

5 bags hold 20 cookies.

B. Subtract those 20 cookies.

Now just 12 are left.

Step 5

Divisor	Work	Groups
4	1 5 2	
	- 4 0	4 × 10
	1 1 2	
	- 8 0	4 × 20
	3 2	
	- 2 0	4 × 5
	1 2	
	- 1 2	4 × 3
	0	

A. With just 12 cookies left, we realize that 3 bags are perfect.

3 x 4 = 12

3 bags hold the last 12 cookies.

B. Subtract the final 12 cookies.

No cookies are left.

Step 6

Divisor	Work	Groups	Sum
4	1 5 2		10
	- 4 0	4 × 10	20
	1 1 2		5
	- 8 0	4 × 20	+ 3
	3 2		38
	- 2 0	4 × 5	
	1 2		
	- 1 2	4 × 3	
	0		

Add up the groups (bags) we made to find the quotient (answer).

Answer: 38

152 ÷ 4 = 38

Tip: There is always more than one path to the solution, and any accurate path is acceptable.

Turn to the following page to see the above problem solved 3 more ways using partial quotients.

Partial Quotients (Continued)

Solve: 152 ÷ 4

In Context: Parents made 152 cookies for a bake sale at school. They are to be packed into bags that are each filled with 4 cookies. How many bags of cookies will be filled for the parents to sell?

Here is the same problem, 152 ÷ 4, solved three more ways:

This student used a very inefficient approach with many steps, but still successfully solved the problem.

4) 1 5 2		
- 4 0	4 × 10	
1 1 2		
- 4 0	4 × 10	10
7 2		10
- 4 0	4 × 10	10
3 2		1
- 4	4 × 1	1
2 8		1
- 4	4 × 1	1
2 4		1
- 4	4 × 1	1
2 0		1
- 4	4 × 1	+ 1
1 6		38
- 4	4 × 1	
1 2		
- 4	4 × 1	
8		
- 4	4 × 1	
4		
- 4	4 × 1	
0		

This student used a more efficient approach and completed the problem in just 4 steps.

4) 1 5 2		
- 8 0	4 × 20	
7 2		20
- 4 0	4 × 10	10
3 2		4
- 1 6	4 × 4	+ 4
1 6		38
- 1 6	4 × 4	
0		

This student used a very efficient approach that includes the same 2 steps that someone would use in long division.

4) 1 5 2		
- 1 2 0	4 × 30	30
3 2		+ 8
- 3 2	4 × 8	38
0		

You may be wondering: "Why partial quotients instead of long division?"

When solving $4\overline{)152}$ with long division, the first thing you ask yourself is, "How many fours are in 15?" This makes no mathematical sense. It's not 15. It is 150. Then you subtract a 12, which is really 120. This is why long division is so hard for many students to understand, and some students never really understand what the steps mean. ***Partial quotients can be just as efficient once mastered***, but they also show the true value of the numbers throughout the problem. Long division is introduced in Grade 6.

Tips:

- Encourage students who may be only multiplying by 1, 2, or 10 to begin trying other factors as they are ready.
- As students begin to understand how to use multiplication and division facts and estimation to their advantage, this method can become very efficient, as seen in the last example above.
- Long division is not introduced until 6th grade.

Multiplying Up

Multiplication and division are opposite operations so students can practice estimation and use multiplication skills to solve division problems with the “multiplying up” strategy. Here groups are multiplied together, adding more and more groups until the total is reached.

Solve: 400 ÷ 16

In Context: The public library received a donation of $400 for new books. They want to order as many books as they can. If the books are $16 each, how many books could they order?

Steps: To multiply up, ask yourself, “How many groups of 16 dollars can I make, that will get me as close to $400 as possible without going over?” Continue making more groups of $16 until we reach $400.

First, we will multiply to make groups of 16 dollars, trying get to 400 total. It helps to imagine you have already spent that money on books.

Each time we multiply to make the groups of 16 dollars to buy books, we will add that amount to keep track of how much we have spent.

1. Start multiplying by 10.   16 x 10 = 160
10 books cost 160 dollars.

2. Start with $400, so write down the 160 dollars spent.

3. Multiply by 10 again to buy 10 more books.
16 x 10 = 160

4. Another 160 dollars was spent. Add this up.
160 + 160 = $320 spent.

5. With 80 dollars left, how many books can we buy? Try 2 more books.
Multiply by 2.   16 x 2 = 32

6. Add 32 dollars.
320 + 52 = $352 spent.

7. Try 2 more books.
Multiply by 2.   16 x 2 = 32

8. Add another 32 dollars.
352 + 32 = $384 spent.

9. With 16 dollars left, buy 1 more book. 16 x 1 = 16

10. Add another $16.
384 + 16 = 400
Perfect! We spent $400.

400 ÷ 16

16 x 10 = 160	**160**
16 x 10 = 160	**+ 160**
	320
16 x 2 = 32	**+ 32**
	352
16 x 2 = 32	**+ 32**
	384
16 x 1 = 16	**+ 16**
	400

10 + 10 + 2 + 2 + 1 = 25

11. All of the money is spent. Add the total groups of 16 combined to make 400 (total number of books).
10 + 10 + 2 + 2 + 1 = 25   They can buy **25** books at $16 each.   **Answer: $400 ÷ $16 = 25 books.**

Tip: There is always more than one path to the solution, and any accurate path is acceptable.

Turn to the following page to see the above problem solved four more ways by multiplying up.

Multiplying Up (Continued)

Solve: 400 ÷ 16

In Context: The public library received a donation of $400 for new books. They want to order as many books as they can. If the books are $16 each, how many books could they order?

Here is the same problem, 400 ÷ 16, solved four more ways:

This is a relatively inefficient approach with many steps, but it is still correct.

400 ÷ 16

16 x 10 = 160	160
16 x 10 = 160	+ 160
	320
16 x 1 = 16	+ 16
	336
16 x 1 = 16	+ 16
	352
16 x 1 = 16	+ 16
	368
16 x 1 = 16	+ 16
	384
16 x 1 = 16	+ 16
	400

10 + 10 + 1 + 1 + 1 + 1 + 1 = 25

This is a more efficient way than the previous example.

400 ÷ 16

16 x 10 = 160	160
16 x 10 = 160	+ 160
	320
16 x 5 = 80	+ 80
	400

10 + 10 + 5 = 25

This is a <u>very efficient</u> approach that includes the same two steps used in long division.

400 ÷ 16

16 x 20 = 320	320
16 x 5 = 80	+ 80
	400

20 + 5 = 25

An acceptable alternative approach is to **subtract** from the total and move toward 0, instead of adding up from 0 to reach the total.

400 ÷ 16

	400
16 x 10 = 160	- 160
	240
16 x 10 = 160	- 160
	80
16 x 2 = 32	- 32
	48
16 x 2 = 32	- 32
	16
16 x 1 = 16	- 16
	0

10 + 10 + 2 + 2 + 1 = 25

Tips:

- Encourage students who may be multiplying by only 1, 2, or 10 to begin trying other factors as they are ready.
- Remember that this approach is not about solving efficiently at first. It is about deepening students' understanding and mathematical flexibility.
- As students begin to understand how to use multiplication and division facts, as well as estimation, this method can become much more efficient, as seen in the bottom middle example above.

Area Model for Division

You can use the area model to solve division problems.

This works because: length × width = area of a rectangle, so…

area ÷ length = width **or** **area ÷ width = length**

Using graph paper for this strategy is strongly recommended.

Example #1: Solve: 288 ÷ 12

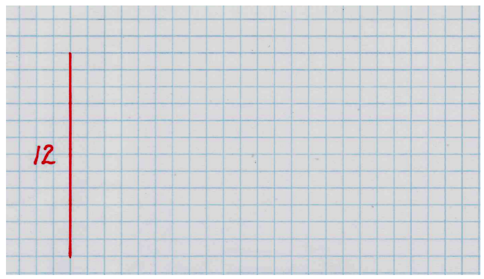
In Context: Farmer Brown wants to have 288 square yards of fenced area for his pigs. If he builds a rectangular fence with a width of 12 yards, how long should he make the fence?

Note: You don't need an area problem to use this model, but one was used here as an example.

Steps: If area = 12 x $\boxed{}$ = 288,

then... 288 ÷ 12 = $\boxed{}$

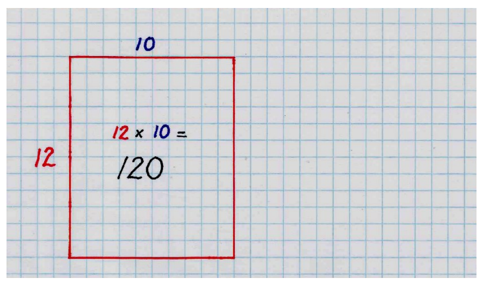
1. The area is 288, and one side of the rectangle is 12. Trace and label one side so it is 12 units long, near the edge of a sheet of graph paper.



288 ÷ 12

The other dimension is still unknown. Finding it will provide the answer to 288 ÷ 12.

2. Start grouping columns of 12 by drawing small rectangular chunks, adding these up each time until a total area of 288 is reached.
Start by drawing a section to show 12 x 10 = 120. Write 120 off to the side to start adding up the sections.

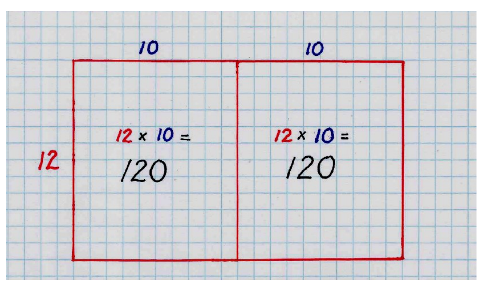


288 ÷ 12

12 x 10 = 120 **1 2 0**

3. Draw another section to show another 12 x 10 = 120.

Add the two areas found so far:
120 + 120 = 240.



288 ÷ 12

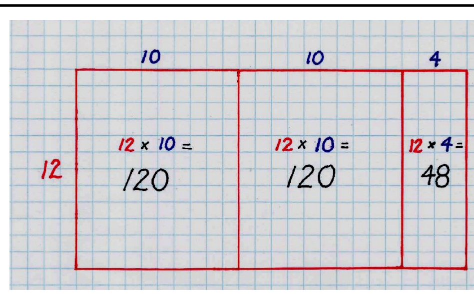
Add:

12 x 10 = 120 **1 2 0**
12 x 10 = 120 **+ 1 2 0**
2 4 0

4. Think: How much more area is needed to reach 288? If needed, refer to the grid for help.

Try 4 more columns of 12.

Draw another section to show 12 x 4 = 48.



288 ÷ 12

Add:

12 x 10 = 120 **1 2 0**
12 x 10 = 120 **+ 1 2 0**
2 4 0
12 x 4 = 48 **+ 4 8**
2 8 8

10 + 10 + 4 = 24

Add the areas 120 + 120 + 48 = 288. We have reached the total area of 288.

5. To find the answer, add the number of columns of 12 it took to make 288 : 10 + 10 + 4 = 24.
12 x 24 = 288, so **288 ÷ 12 = 24**. The length of the fence is 24 yds.